- M is orientable $\Longrightarrow$ $\exists$ a volume form.

② For an orientable mfd M^n, the volume w.r.t. a volume form $\Omega \in \Omega^n(M)$ is defined by $\int_M \Omega$.

- If furthermore M is cpt, then $\int_M \Omega < +\infty$.

Rmk If $\Omega \in \Omega^n(M)$ is a volume form, then $f \cdot \Omega$ is also a volume form if f is a nowhere vanishing function.

Rmk To some extent, $\mathrm{div}(X)$ detects how the volume w.r.t Ω changes.

More precisely,

$$\underset{\substack{\cap \\ M}}{\int_D} \mathrm{div}(X)\,\Omega = \int_D \mathcal{L}_X \Omega \overset{\text{by def}}{=} \int_D \frac{d}{dt}\Big|_{t=0} (\varphi_X^t)^* \Omega$$

$$= \frac{d}{dt}\Big|_{t=0} \int_D (\varphi_X^t)^* \Omega = \frac{d}{dt}\Big|_{t=0} \int_{(\varphi_X^t)(D)} \Omega .$$

6. Applications

- For a cpt mfd M^n, the space $\Omega^n(M)$ is an ∞-ly dim'l vector space over $\mathbb{R}$ (b/c $\Omega^n(M) \simeq C^\infty(M;\mathbb{R})$).

also ∞-ly dim'l vector space over $\mathbb{R}$

$$\Omega^n(M) \supseteq \{ d\theta \mid \theta \in \Omega^{n-1}(M) \} =: \underbrace{\mathcal{E}^n(M)}_{\text{exact } n\text{-form}}$$

Consider $\Omega^n(M) / \mathcal{E}^n(M) =: H^n(M;\mathbb{R})$

Thm (de Rham) $H^n(M^n;\mathbb{R})$ is finite dim'l over $\mathbb{R}$.

Prop: If M^n is cpt, connected, without b/d and orientable, then for any class $\alpha = [\theta] \in H^n(M^n;\mathbb{R})$, the integration

$$\alpha = [\theta] \longmapsto \int_M \theta \qquad (*)$$

is well-defined and surjective to $\mathbb{R}$. Moreover, $(*)$ is a homomorphism with $\ker = \{0\} \in H^n(M^n;\mathbb{R})$. In particular, $\dim_{\mathbb{R}} H^n(M^n;\mathbb{R}) = 1$.

Pf ① For $\theta + d\sigma$ (so $[\theta + d\sigma] = \alpha$), we have

$$\int_M \theta + d\sigma = \int_M \theta + \int_M d\sigma = \int_M \theta + \cancel{\int_{\partial M} \overset{0}{\sigma}} = \int_M \theta.$$

b/c $\partial M = \phi$

② Since M is orientable, $\exists$ nowhere vanishing n-form $\Omega \in \Omega^n(M)$. In particular, $\Omega \underset{\text{locally}}{=} f\, dx_1 \wedge \cdots \wedge dx_n$ with $f > 0$ for each local chart, so $\int_M \Omega > 0 \in \mathbb{R}$. Hence $(*)$ maps onto $\mathbb{R}$ by rescaling Ω.

③ $(*)$ is obviously a homomorphism. It suffices to show if $\alpha = [\theta] \longmapsto \int_M \theta = 0$, then $\theta = d\sigma$ for some $\sigma \in \Omega^{n-1}(M)$. The proof is given by local argument and induction.

$$\int_{(0,1)} \underbrace{f(x)dx}_{\substack{\text{1-form} \\ \text{cpt supp in } (0,1)}} = 0 \implies \exists \overset{\text{0-form cc } (0,1)}{\tilde{g}(x)} \text{ s.t. } dg(x) = f(x)dx$$

simply set $g(x) = \int_0^x f(t)dt$ ∎

<u>Rmk</u> (Fact) $H^n(M^n;\mathbb{R}) \neq 0$ iff M^n is orientable

<u>Rmk</u> All the argument works in the same way for non-cpt mfd or mfd with b/d, simply consider cpt supp n-form.

In fact, $H^n(M^n;\mathbb{R})$ is called the n-th (or top) de Rham cohomology group, usually denoted by $H^n_{dR}(M^n;\mathbb{R})$. For a general de Rham cohomology theory, see next Lecture

<u>Prop</u> (Moser's trick). Let M^n be a cpt. connected mfd without b/d, and $\Omega_0, \Omega_1 \in \Omega^n(M)$ be two volume forms. Then

$$\int_M \Omega_0 = \int_M \Omega_1 \iff \exists\, \varphi : M \circlearrowleft \text{ s.t } \varphi^* \Omega_1 = \Omega_0$$

Pf "$\Leftarrow$": base change formula

"$\Rightarrow$": $\displaystyle\int_M \Omega_0 - \Omega_1 = 0 \underset{\text{above}}{\overset{\text{by Prop}}{\rightsquigarrow}} \Omega_0 - \Omega_1 = d\sigma$ for $\sigma \in \Omega^{n-1}(M)$.

Consider $\Omega_t = t\Omega_1 + (1-t)\Omega_0 \quad t\in[0,1]$. Then Ω_t is a volume form for every $t\in[0,1]$, so $\exists$ vector fields X_t s.t.

$$\iota_{X_t}\Omega_t = \Omega_t(X_t, \cdots) = \sigma$$

Denote by φ_X^t the 1-par group of diffeo generated by X_t. ← time-dependent

$$\frac{d((\varphi_X^t)^*\theta)}{dt} = (\varphi_X^t)^* \mathcal{L}_X\theta$$

Then

$$\frac{d((\varphi_X^t)^*\Omega_t)}{dt} = (\varphi_X^t)^*(\mathcal{L}_{X_t}\Omega_t) + (\varphi_X^t)^*\left(\frac{d\Omega_t}{dt}\right)$$

$$= (\varphi_X^t)^*(d\,\iota_{X_t}\Omega_t) + (\varphi_X^t)^*(\Omega_1-\Omega_0)$$

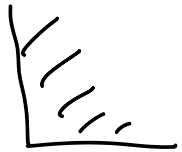
$$= (\varphi_X^t)^* d\sigma + (\varphi_X^t)^*(-d\sigma) = 0$$

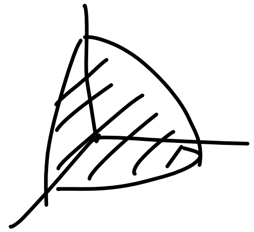
$\Rightarrow (\varphi_X^t)^*\Omega_t = \text{constant}$

$\Rightarrow (\varphi_X^1)^*\Omega_1 = \Omega_0$

$\square$

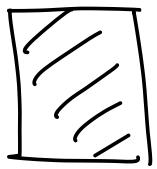
mfd with corner: locally it looks like "corner"





$\cdots$ local model

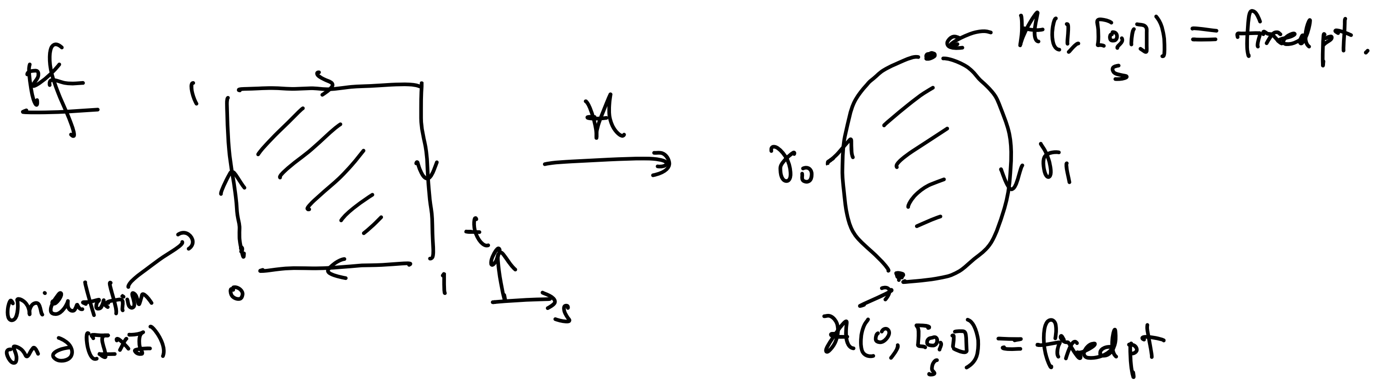
one extends the definition of mfd with b/d to mfd with corner.

e.g. $\overset{I}{[0,1]} \overset{\times}{\times} \overset{I}{[0,1]} =$  $\xleftarrow{\text{recall}}$ $\partial(M \times N) = \partial M \times N \cup M \times \partial N$

<u>Thm</u> Stokes' Thm holds for mfd with corner.

<u>Cor</u> M smooth mfd, $\gamma_0, \gamma_1 : [0,1] \to M$ are path-homotopic smooth curve with endpts fixed. Then for any <u>closed</u> 1-form $\theta \in \Omega^1(M)$, we have $\int_{\gamma_0} \theta = \int_{\gamma_1} \theta$.

pf



Then for $\theta \in \Omega^1(M)$,

$$0 = \int_{H(I\times I)} d\theta = \int_{I\times I} H^*(d\theta) \implies 0 = \int_{I\times I} d(H^*\theta)$$

$$\overset{\text{stokes's thm}}{=} \int_{\partial(I\times I)} H^*\theta$$

$$= \int_{H(\partial(I\times I))} \theta$$

$$= \underbrace{\int_{\gamma_0} \theta}_{\text{positive orientation}} \underbrace{- \int_{\gamma_1} \theta}_{\text{negative orientation}}$$

$\square$

Rmk The same argument works for any piecewise smooth curves γ_0 and γ_1 that are htp with end pt fixed.

Cor If M is a simply connected mfd, then any closed

$\pi_1(M)=0$

1-form θ is exact, i.e. $\theta = df$ for some fcn $f \in C^\infty(M;\mathbb{R})$.

pf By top assumption, any loop $\gamma \subset M$ contracts to the base pt. therefore by Cor above,

$$\int_\gamma \theta = \int_{pt} \theta = 0$$

Then define $f: M \longrightarrow \mathbb{R}$ $\quad x \longmapsto f(x) := \int_{x_0}^{x} \theta$ where x_0 is any fixed base pt in M and $\int_{x_0}^{x} \theta$ means $\int_{\tilde{\gamma}\text{ any path connecting } x_0 \text{ to } x} \theta$

Note that this is well-defined and independent of $\tilde{\gamma}$.

One can check that $df = \theta$.

(cf. proof of the explicit prop of deRham Thm above).

Rmk On $\mathbb{R}^2\setminus\{0\}$, consider 1-form

$$\theta = \frac{-y}{x^2+y^2}dx + \frac{x}{x^2+y^2}dy$$

Then
$$d\theta = \frac{-(x^2+y^2)+2y^2}{x^2+y^2}dy\wedge dx + \frac{(x^2+y^2)-2x^2}{x^2+y^2}dx\wedge dy$$
$$= 0$$

Therefore, θ is a closed 1-form on $\mathbb{R}^2\setminus\{0\}$.

Claim: θ is NOT exact.

If so, $\theta = df$, then consider circle $\gamma = \gamma(t) = (\cos t, \sin t)$ for $t \in [0, 2\pi)$. then

- $\int_\gamma \theta \underset{\substack{\uparrow \\ \text{Stokes}}}{=} \int_\gamma df = \int_{\partial\gamma} f = 0$

- $\int_\gamma \theta \underset{\substack{\uparrow \\ \text{parametrization}}}{=} \int_0^{2\pi} (-\sin t)(-\sin t)\,dt + \cos t \cdot \cos t \cdot dt = \int_0^{2\pi} 1\,dt = 2\pi$

$\rightarrow\leftarrow$

This indicates that "simply connected" condition in the previous cor can not be removed.

This also indicates that difference between closed forms and exact forms can detect topology (e.g. $\mathbb{R}^2 \setminus \{0\} \simeq S^1$).

Informally:

$$\text{Top} \longleftrightarrow \text{closed forms} / \text{exact forms}$$

This is more or less what de Rham theory says (see later lecture).